The Borel dichromatic number

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Caltech Logic Seminar
April 29, 2026

Outline

1. **Background:** Borel chromatic numbers and basis problems
2. **Dichromatic number:** a directed analogue
3. **Classical reduction:** from hypergraphs to digraphs
4. **Borel lifting:** Thornton's coding framework
5. **Open questions**

Borel graphs

A Borel graph $G = (X, E)$ consists of:

- a standard Borel space X (vertex set),
- a Borel symmetric irreflexive relation $E \subseteq X^2$ (edges).

A Borel k-coloring is a Borel map $c\colon X \to k$ with $c(x) \neq c(y)$ for all $(x, y) \in E$.

The Borel chromatic number $\chi_B(G)$ is the minimum k admitting a Borel k-coloring.

Basic fact for finite graphs; proved in KST '99 in the Borel setting

Every graph of degree $\leq d$ admits a $(d + 1)$-coloring.

Why Borel?

Every Borel graph has a (possibly non-Borel) proper coloring.

The point: requiring the coloring to be Borel makes the problem genuinely harder.

Example 1

There exist Borel graphs with $\chi(G) = 2$ but $\chi_B(G) > \aleph_0$.

The Borel chromatic number captures a notion of *definable complexity* of the coloring problem.

The $\mathbb{G}_0$-dichotomy (Kechris–Solecki–Todorčević, 1999)

Theorem 2 (KST '99)

There exists a Borel graph $\mathbb{G}_0$ such that, for every analytic graph G:

$$\chi_B(G) > \aleph_0 \quad \Longleftrightarrow \quad \mathbb{G}_0 \leq_B G$$

where $\leq_B$ denotes the existence of a Borel homomorphism.

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In other words: $\{\mathbb{G}_0\}$ is a one-element basis for the class of Borel graphs with uncountable Borel chromatic number.

Interpretation: To prove $\chi_B(G) > \aleph_0$, there is essentially *one* obstruction to find. The problem is "well-structured."

What is a basis?

Fix a class $\mathscr{C}$ of Borel graphs (e.g. those with $\chi_B \geq k$).

A family $\mathbf{B}$ is a basis for $\mathscr{C}$ under Borel homomorphism if:

$$G \in \mathscr{C} \quad \iff \quad \exists\, H \in \mathbf{B},\ H \leq_B G.$$

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Small basis = "simple" decision problem:

- Size 1: one obstruction to check (like $\mathbb{G}_0$)
- Finite: finitely many obstructions
- Countable: "tame" complexity
- Uncountable or none: the problem is genuinely complex

The $\mathbb{L}_0$-dichotomy

Theorem 3 (Carroy–Miller–Schrittesser–Vidnyánszky)

There exists a Borel graph $\mathbb{L}_0$ *such that, for every analytic graph* G*:*

$$\chi_B(G) > 2 \quad \Longleftrightarrow \quad \mathbb{L}_0 \leq_B G.$$

Again a one-element basis, this time for the 2 vs. 3-color threshold.

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So far: at both the $2/3$-color threshold and the countable/uncountable threshold, there is a dichotomy with a one-element basis.

Question: Does this continue for higher thresholds?

No dichotomy at higher thresholds

Theorem 4 (Todorčević–Vidnyánszky, 2021)

For every $k \geq 3$, the set of (codes for) Borel graphs admitting a Borel k-coloring is $\boldsymbol{\Sigma}^1_2$-complete.

No dichotomy at higher thresholds

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Corollary 5

For $k \geq 3$: no countable basis for the class of Borel graphs with $\chi_B(G) > k$.

The landscape for *undirected* Borel chromatic numbers is completely understood:

$\chi_B > 2$	basis of size 1
$\chi_B > k$, $k \geq 3$	no countable basis
$\chi_B > \aleph_0$	basis of size 1

Interlude: what does $\boldsymbol{\Sigma}^1_2$-completeness mean?

Recall the projective hierarchy:

$$\begin{tikzcd}
\Sigma^1_2 & & \Pi^1_2 \\
& \Delta^1_2 \arrow[ul, hook] \arrow[ur, hook] & \\
\Sigma^1_1 \text{ (analytic)} \arrow[ur, hook] & & \Pi^1_1 \text{ (coanalytic)} \arrow[ul, hook] \\
& \Delta^1_1 \text{ (Borel)} \arrow[ul, hook] \arrow[ur, hook] &
\end{tikzcd}$$

A set $A \subseteq \omega$ is Σ^1_2 if there is a Π^1_1 set $P \subseteq \omega \times \mathcal{N}$ with:

$$n \in A \quad \Longleftrightarrow \quad \exists x \in \mathcal{N},\ (n, x) \in P.$$

Σ^1_2-completeness: the key example

Why “admits a Borel k-coloring” is Σ^1_2

Given a *nice code* e_G for a Borel graph G:

$$\text{“}G\text{ admits a Borel }k\text{-coloring”} \iff \exists\, e_c \in \mathcal{N}\ \Big[\underbrace{\text{“}\mathcal{D}_{e_c}\text{ is a }k\text{-coloring of }G\text{”}}_{\Pi^1_1 \text{ in } e_c, e_G}\Big]$$

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The inner condition is Π^1_1 because it asserts:

- totality: $\forall x \in V(G),\ \exists! i < k,\ c(x) = i \quad [\Pi^1_1]$
- properness: $\forall (x, y) \in E(G),\ c(x) \neq c(y) \quad [\Pi^1_1]$

The outer $\exists e_c$ quantifier makes the whole thing Σ^1_2.

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The outer $\exists e_c$ quantifier makes the whole thing Σ^1_2.

Σ^1_2-complete means: this set is *as hard as any* Σ^1_2 set. Every Σ^1_2 set reduces to it via a Δ^1_1 function.

Why does Σ^1_2-completeness destroy countable bases?

Proof sketch (for chromatic number).

Suppose $\mathbf{B} = \{H_1, H_2, \ldots\}$ is a countable basis for $\chi_B(G) > k$. Then:

$$\chi_B(G) > k \iff \exists\, n,\ \exists \text{ Borel homomorphism } H_n \to G.$$

For each fixed H_n, "$\exists$ Borel hom $H_n \to G$" is Σ^1_2 in the code for G.

A countable union of Σ^1_2 sets is Σ^1_2.

But the left-hand side $\{\text{code}(G) : \chi_B(G) > k\}$ is Π^1_2-complete.

This gives $\Pi^1_2 \subseteq \Sigma^1_2$, contradicting the strictness of the projective hierarchy. □

The dichromatic number (Neumann-Lara, 1982)

Let $D = (V, A)$ be a directed graph.

A dicoloring is a map $c\colon V \to k$ such that every color class induces an *acyclic* subdigraph (no directed cycles).

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Example 6

- A tournament on 3 vertices forming a directed 3-cycle: $\vec{\chi} = 2$.
- A directed graph with no directed cycles: $\vec{\chi} = 1$.
- If D is a symmetric digraph ($\{u, v\} \in E \Leftrightarrow (u, v), (v, u) \in A$): $\vec{\chi}(D) = \chi$(underlying graph).

Dichromatic vs. chromatic: key difference

Classical complexity

- “Is $\chi(G) \leq 2$?” is decidable in polynomial time (bipartiteness).
- “Is $\vec{\chi}(D) \leq 2$?” is NP-complete (Bokal et al., 2004).

The dichromatic number is “harder” than the chromatic number: complexity starts already at the $2/3$-color threshold.

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Key observation: Acyclicity is *not* a local constraint.
Checking that a color class is independent (for chromatic number) requires looking at edges one at a time.
Checking that a color class is acyclic may require following long directed paths.

In particular, dicoloring is not a CSP since acyclicity cannot be expressed as a homomorphism to a fixed finite template.

Borel dichromatic number

For Borel directed graphs, define $\vec{\chi}_B(D)$ analogously, requiring the dicoloring to be Borel.

Theorem 7 (Raghavan–Xiao, 2024)

There exists a family $\{H_\alpha\}_{\alpha<\mathfrak{c}}$ *of Borel digraphs, of size continuum, such that:*

$$\vec{\chi}_B(D) > \aleph_0 \quad \Longleftrightarrow \quad \exists\, \alpha,\ H_\alpha \leq_B D.$$

The H_α *are pairwise Borel-incomparable.*

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The H_α are pairwise Borel-incomparable.

This is the directed analogue of the $\mathbb{G}_0$-dichotomy, but the basis has *continuum* many elements instead of one.

Open: Is $\mathfrak{c}$ optimal, or could a smaller (perhaps countable) basis exist?

The main result

Proposition 8

The set of nice codes for Borel directed graphs admitting a Borel 2-dicoloring is $\mathbf{\Sigma}^1_2$-complete.
Equivalently, the set of nice codes for Borel directed graphs with $\vec{\chi}_B(D) \geq 3$ is $\mathbf{\Pi}^1_2$-complete.

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Corollary 9

There is no countable basis for Borel digraphs of Borel dichromatic number ≥ 3*.*

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Corollary 9

There is no countable basis for Borel digraphs of Borel dichromatic number ≥ 3.

Recovering the chromatic number

Given an undirected graph G, symmetrize it: $\vec{G}$ has arcs (u, v) and (v, u) for each edge $\{u, v\}$. Then $\vec{\chi}_B(\vec{G}) = \chi_B(G)$.
This reduction is Δ^1_1 on codes, so Σ^1_2-completeness of k-dicoloring for $k \geq 4$ follows from Todorčević–Vidnyánszky.

The complete picture

Class	**Basis?**	**Reference**
$\chi_B(G) > \aleph_0$	Yes, size 1 ($\mathbb{G}_0$)	KST '99
$\chi_B(G) > 2$	Yes, size 1 ($\mathbb{L}_0$)	CMSV
$\chi_B(G) > k$, $k \geq 3$	No ($\mathbf{\Pi}^1_2$-complete)	TV '21
$\vec{\chi}_B(D) > \aleph_0$	Yes, size $\mathfrak{c}$	RX '24
$\vec{\chi}_B(D) > k$, $k \geq 2$	No ($\mathbf{\Pi}^1_2$-complete)	**this talk**

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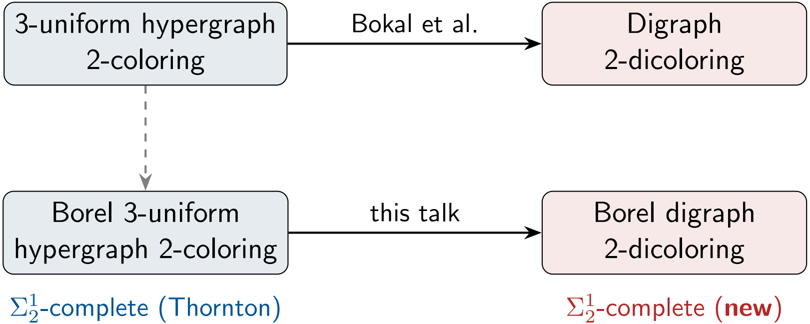
The case $k = 2$ for directed graphs is the last missing piece.

The proof: lift the classical NP-completeness reduction to the Borel setting.

Strategy



3-uniform hypergraph 2-coloring

Bokal et al.

Digraph 2-dicoloring

Borel 3-uniform hypergraph 2-coloring

this talk

Borel digraph 2-dicoloring

Σ^1_2-complete (Thornton)

Σ^1_2-complete (**new**)

Step 1: The classical starting point

Fact 10

The problem of deciding whether a 3-uniform hypergraph admits a 2-coloring is NP-complete.

A 3-uniform hypergraph $H = (V, E)$ has hyperedges $e \in \binom{V}{3}$.
A 2-coloring is a map $c\colon V \to 2$ such that no hyperedge is monochromatic.

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Goal: Reduce this to 2-dicoloring of directed graphs.

Idea: Build a directed graph $D(H)$ from H using a finite *gadget* that translates "no monochromatic hyperedge" into "no monochromatic directed cycle."

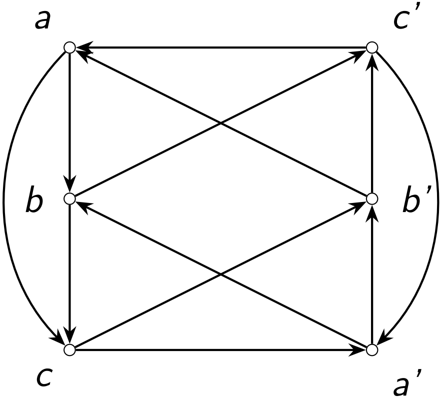
The template gadget

The gadget F has:

- Shared vertices $\{a, b, c\}$ (will become hypergraph vertices)
- Auxiliary vertices $\{a', b', c'\}$ (local to each gadget copy)

Lemma 11 (Gadget property)

A map $\sigma\colon \{a, b, c\} \to 2$ extends to a 2-dicoloring of F iff σ is not constant.



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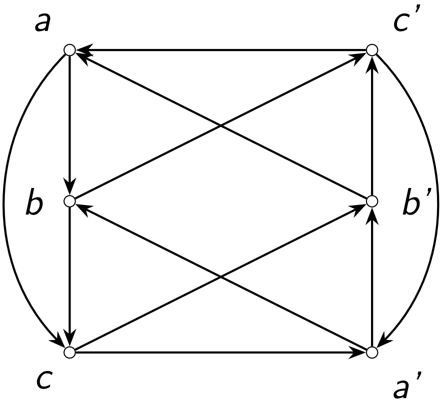

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coloring of $\{a, b, c\}$	extends to F?
$(0,0,1)$, $(0,1,0)$, $(1,0,0)$, $\ldots$	Yes
$(0,0,0)$ or $(1,1,1)$	No

The reduction $H \mapsto D(H)$

Fix a 3-uniform hypergraph $H = (V, E)$ and a linear order on V.

Sorted hyperedges: $E_{\mathrm{sort}} = \{(v_1, v_2, v_3) \in E : v_1 < v_2 < v_3\}$.

Vertex set of $D(H)$:

$$V_D = V \ \cup \ \big\{(e, t) : e \in E_{\mathrm{sort}},\ t \in \{a', b', c'\}\big\}$$

Arc set: For each $e = (v_1, v_2, v_3)$, place a copy F_e of the gadget:

- shared labels $a, b, c \mapsto v_1, v_2, v_3$
- auxiliary labels $a', b', c' \mapsto (e, a'), (e, b'), (e, c')$

Key: Shared vertices are identified across gadgets. Auxiliary vertices are disjoint.

Forward direction: H 2-colorable $\Rightarrow$ $D(H)$ 2-dicolorable

Let $c\colon V \to 2$ be a 2-coloring of H.

For each hyperedge e: the shared vertices get a non-constant coloring, so by the gadget property it extends to a 2-dicoloring of F_e.

Auxiliary vertices of different gadgets are disjoint $\Rightarrow$ the extensions combine into a global 2-dicoloring of $D(H)$.

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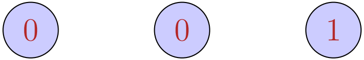

$\Rightarrow$ extends to F_e

non-constant ✓

Reverse direction: $D(H)$ 2-dicolorable $\Rightarrow$ H 2-colorable

Let $\psi\colon V_D \to 2$ be a 2-dicoloring of $D(H)$.

Define $\varphi = \psi|_V$ (just restrict to the original vertices).

For each hyperedge $e = (v_1, v_2, v_3)$:

- F_e is a subdigraph of $D(H)$
- So $\psi|_{V(F_e)}$ is a valid 2-dicoloring of F_e
- By the gadget property: $\varphi(v_1), \varphi(v_2), \varphi(v_3)$ are not all equal

Hence φ is a 2-coloring of H. ☐

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Hence φ is a 2-coloring of H. $\square$

Important observation

The reverse direction is a *plain restriction*. No choice, no selection, no local finiteness needed. This will matter in the Borel setting.

Borel lifting: the goal

Fact 12 (Thornton)

The set of nice codes for locally finite Borel 3-uniform hypergraphs admitting a Borel 2-coloring is $\mathbf{\Sigma}^1_2$*-complete.*

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We need to verify:

1. The map $\mathrm{code}(H) \mapsto \mathrm{code}(D(H))$ is Δ^1_1.
2. "H is Borel 2-colorable" $\iff$ "$D(H)$ is Borel 2-dicolorable" with Borel witnesses.

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Why "Δ^1_1 on codes"?

A Δ^1_1 reduction from a Σ^1_2-complete set preserves Σ^1_2-hardness.
So if the reduction is Δ^1_1 and the equivalence holds, the target is also Σ^1_2-hard.

Thornton's coding framework (brief)

Fix a good ω-parametrization $U \subseteq \omega \times \mathcal{N}$ of Π^1_1.

A nice code for a Δ^1_1 set B is an index e such that B is simultaneously described by a Π^1_1 formula and a Σ^1_1 formula, and these descriptions agree.

The key tool:

Lemma 13 (Toolkit)

The following operations are Δ^1_1 in the codes:

1. $(A, B) \mapsto A \cup B$
2. $(A, B) \mapsto A \cap B$
3. $(A, B) \mapsto A \times B$
4. $A \mapsto \mathrm{dom}(A)$
5. $(A, f) \mapsto f(A)$, f *ctble-to-one*
6. $A \mapsto$ *enum. of finite sections*

Recipe: Express the construction using these operations ⇒ it is Δ^1_1 on codes.

The construction is Δ^1_1 on codes

Let H be a locally finite Borel 3-uniform hypergraph, coded by (V, E). Fix a Δ^1_1 linear order $\preceq$ on $\mathcal{N}$.

Step 1: Sort hyperedges.

$$E_{\text{sort}} = \{(a, b, c) \in E : a \preceq b \preceq c\}$$

intersection with graph of $\preceq$ ⇒ toolkit item (2) ✓

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Step 2: Build vertex set.

$$V_D = V \cup (E_{\text{sort}} \times T_{\text{aux}})$$

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Step 3: Build arc set. (next slide)

Building the arc set

Define the substitution map $\varphi_H \colon E_{\text{sort}} \times E_T \to V_D^2$:

$$(e, (u, v)) \longmapsto \text{(corresponding arc of } D(H)\text{)}$$

- shared label in position $i \mapsto$ vertex v_i from $e = (v_1, v_2, v_3)$ (projection)
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φ_H is Δ^1_1 in the code of H: it uses projections and products, which are recursive.

φ_H is finite-to-one: $|E_T|$ is fixed, and each vertex appears in finitely many hyperedges (local finiteness).

Arc set: $A_D = \varphi_H(E_{\text{sort}} \times E_T)$ ⇒ toolkit item (5) ✓

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Arc set: $A_D = \varphi_H(E_{\text{sort}} \times E_T)$ $\Rightarrow$ toolkit item (5) ✓

Lemma 14

The map $\mathrm{code}(H) \mapsto \mathrm{code}(D(H))$ *is* Δ^1_1.

Borel forward direction: Luzin–Novikov

Claim 15

If H is Borel 2-colorable, then $D(H)$ is Borel 2-dicolorable.

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Proof.

Let $c\colon V \to 2$ be a Borel 2-coloring of H. For each $e \in E_{\mathrm{sort}}$, at least one extension to a 2-dicoloring of F_e exists.

Define the set of valid local completions:

$$S = \{(e, \sigma) \in E_{\mathrm{sort}} \times \mathcal{C}_F : \sigma \text{ extends to a 2-dicoloring of } F_e\}$$

S is Borel, with finite nonempty sections S_e. By the Luzin–Novikov theorem: $\exists$ Borel selector $e \mapsto \sigma(e) \in S_e$.

Pasting these selections with c on shared vertices gives a Borel 2-dicoloring of $D(H)$. □

Borel reverse direction: just restrict

Claim 16

If $D(H)$ is Borel 2-dicolorable, then H is Borel 2-colorable.

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Proof.

Let $\psi\colon V_D \to 2$ be a Borel 2-dicoloring of $D(H)$.

Define $\varphi = \psi|_V$.

Since $V \subseteq V_D$ is Borel, φ is Borel.

For each hyperedge e: $F_e \subseteq D(H)$ is a subdigraph, so $\psi|_{V(F_e)}$ is a valid 2-dicoloring, hence the shared-vertex colors are not all equal.

So φ is a Borel 2-coloring of H. □

Proof of the main result

Proof of Proposition.

Hardness: We have a Δ_1^1 reduction from Borel 2-colorable 3-uniform hypergraphs (Σ_2^1-complete) to Borel 2-dicolorable digraphs. Hence the latter is Σ_2^1-hard.

Proof of the main result

Proof of Proposition.

Hardness: We have a Δ^1_1 reduction from Borel 2-colorable 3-uniform hypergraphs (Σ^1_2-complete) to Borel 2-dicolorable digraphs. Hence the latter is Σ^1_2-hard.

Upper bound:

$$\text{“}D\text{ admits a Borel 2-dicoloring”} \quad\Longleftrightarrow\quad \exists\, e_c\, \big[\underbrace{\text{“}\mathcal{D}_{e_c}\text{ is a valid 2-dicoloring”}}_{\Pi^1_1}\big]$$

The Π^1_1 condition checks:

- totality and range $\subseteq \{0,1\}$ $\quad(\Pi^1_1)$
- each color class induces no directed cycle $\quad(\Pi^1_1\text{: universal over finite paths})$

So the set is Σ^1_2. Combined with hardness: Σ^1_2-complete. ☐

The corollary: no countable basis

Proof of Corollary.

Suppose $\mathbf{B} = \{H_1, H_2, \dots\}$ is a countable basis for $\vec{\chi}_B(D) \geq 3$:

$$\vec{\chi}_B(D) \geq 3 \quad \Longleftrightarrow \quad \exists\, n,\ \exists \text{ Borel hom } H_n \to D.$$

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For fixed H_n: "$\exists$ Borel hom $H_n \to D$" is Σ^1_2 in the code of D.

Countable union $\Rightarrow$ the right-hand side is Σ^1_2.

But the left-hand side is Π^1_2-complete (complement of our Σ^1_2-complete set).

$\Pi^1_2 \subseteq \Sigma^1_2$ contradicts the strictness of the projective hierarchy. $\square$ $\square$

A stronger conclusion

The complexity argument rules out more than just a countable basis.

Observation

If $\mathbf{B} \in \Sigma^1_2$ as a subset of the code space, then:

- "$H \in \mathbf{B}$" is Σ^1_2
- "$\exists$ Borel hom $H \to D$" is Σ^1_2
- conjunction and $\exists$-quantification preserve Σ^1_2

So the right-hand side would be Σ^1_2, contradicting Π^1_2-completeness.

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So the right-hand side would be Σ^1_2, contradicting Π^1_2-completeness.

Conclusion: Any basis $\mathbf{B}$ must satisfy $\mathbf{B} \notin \Sigma^1_2$. It is at least as complex to describe as the set of all digraphs without a Borel acyclic 2-coloring.

A complexity lower bound destroys all "simple" bases, not just countable ones.

Open questions I: the uncountable threshold

Raghavan–Xiao give a continuum-size basis for $\vec{\chi}_B(D) > \aleph_0$.

Q1. **Is the basis irredundant?** I.e., does removing any single element break the basis? (Likely yes, from pairwise incomparability.)

Q2. **Can the basis be made smaller?** Is $\mathfrak{c}$ optimal?

Q3. **Is there a direct (non-complexity) proof** that no *countable* basis exists for $\vec{\chi}_B(D) > \aleph_0$? Perhaps via diagonalization, adapting Miller's anti-basis argument for G_f graphs.

Note: the complexity approach does *not* work here—the continuum-size basis shows the set is not Π^1_2-complete.

Open questions II: structure and CSPs

Q4. Structural restrictions. Does a countable basis exist for digraphs G_f (generated by a single Borel function) with $\vec{\chi}_B(G_f) \geq 3$?
Miller showed no countable basis exists for undirected G_f with $\chi_B \geq 3$.

Q5. Bounded width CSPs. Thornton showed that width 1 structures are exactly those where solvable Borel instances have Borel solutions.
Grebík–Vidnyánszky showed the Borel/classical CSP dichotomies diverge.
What is the exact complexity dividing line within bounded width?

Q6. Measurable arboricity (Thornton). Arboricity = undirected analogue of dichromatic number.
Can μ-measurable arboricity differ from classical arboricity by more than 1?

Thank you.

`arXiv/2604.05228`