A Concise Proof of the L_0 Dichotomy

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Abstract

Carroy, Miller, Schrittesser, and Vidnyánszky established the L_0 dichotomy: there is a Borel graph of Borel chromatic number three that admits a continuous homomorphism to every analytic graph of Borel chromatic number at least three. Their proof relies on a transfinite analysis of terminal approximations over a decreasing ω_1-sequence of analytic sets.

This brief note offers a new, substantially shorter proof of this result by adapting the graph-theoretic framework recently introduced by Bernshteyn for the G_0 dichotomy. The central device is a σ-ideal of small sets of homomorphisms from finite path approximations into the target graph, where smallness is witnessed by a bounded odd-walk condition on vertex projections. The key lemma that largeness is preserved under the doubling operation is established via the First Reflection Theorem, replacing the original transfinite construction with a single Borel reflection argument. The continuous homomorphism from the canonical graph $\mathbb{L}_c$ into the target is then obtained as a limit of shrinking families of copies, in direct analogy with Bernshteyn's proof for G_0.

1 Introduction

Throughout this entire work, G denotes an analytic graph on a Polish space $V(G) = X$. That is, X is a separable completely metrizable topological space, and G is a symmetric and irreflexive relation on X which, as a subspace of $X \times X$, is the continuous image of a Polish space.

Denote the *Borel chromatic number* of G by $\chi_B(G)$, that is, $\chi_B(G)$ is the least number of colours k needed to find a Borel mapping $c\colon X \to k$ in such a way that adjacent vertices in G are given different colours. We employ the symbol $H \to_c G$ (respectively, $H \to_B G$) to abbreviate the fact that there is a continuous (respectively Borel) homomorphism from the analytic graph H to the analytic graph G. It is straightforward to show the following.

Fact 1.1. *If $H \to_c G$ or, more generally if $H \to_B G$, then $\chi_B(H) \leq \chi_B(G)$.*

Using a standard greedy algorithm argument, it is clear that any finite graph G of maximum degree Δ satisfies $\chi(G) \leq \Delta + 1$. Interestingly, a similar fact holds for Borel graphs of uniformly bounded degree.

Theorem 1.2 (Proposition 4.6 in [KST99]). *If G is a Borel graph on a Polish space all of whose degrees are bounded by the natural number k, then $\chi_B(G) \leq k+1$.*

The following notion, taken from [Kec95, Theorem 35.10, p. 285], is a well-known fact about analytic sets.

Definition 1.3. A collection of subsets Φ of a Polish space X is said to be Π^1_1 *on* Σ^1_1 if for any Polish Y and any Σ^1_1 set $A \subseteq Y \times X$, the set $A_\Phi := \{y \in Y : A_y := \{x \in X : (y,x) \in A\} \in \Phi\}$ is Π^1_1.

Lemma 1.4 (First Reflection Theorem). *If Φ is as in the preceding definition, then for any Σ^1_1 set $A \in \Phi$, there is a Borel set $B \subseteq X$ such that $A \subseteq B$ and $B \in \Phi$.*

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2 Finite path approximations

Fix the parameter $c \in \omega^\omega$ and a path of length ω on the vertex set $\{p_n\}_{n<\omega}$. That is, the edge set $\{(p_n, p_{n+1}) : n < \omega\}$. Naturally, one could assume that $p_n = n$, but the labeling helps distinguish these vertices from other indices.

Recall that if s and t are finite strings (of, say, natural numbers), then $s^\frown t$ is the *concatenation* of s and t. Similarly, t^n is the repeated concatenation of t with itself, and (i) denotes the string of single value i. Next, recursively construct a family of graphs $\{L^c_n : n < \omega\}$ such that for all $n < \omega$: L^c_n is a finite path; if $n > 0$, the endpoints of L^c_n are $e^n_i := (p_0)^\frown(0)^{n-1\frown}(i)$, for $i < 2$; and

$$\text{Every vertex of } L^c_n \text{ is of the form } (p_k)^\frown t \text{ for some } k \text{ and } t \in 2^{\leq n}. \tag{1}$$

Unless $t = \emptyset$, call such vertex a *non-path vertex*, and call the vertices of the form (p_k) the *path vertices* of L^c_n.

Start with L^c_0 as the graph consisting of a single vertex p_0 and no edges, and set $e^0_0 = e^0_1 = p_0$. Now suppose that L^c_n has been constructed. L^c_{n+1} will be a path obtained from joining two copies of L^c_n by a path of $c(n)+2$ edges (with $c(n)+1$ *new* path vertices), but one must update the vertex labels in order to distinguish the two copies. One accomplishes this by simply appending 0 to each vertex of the first copy, and 1 to the other. Formally, start by adding the vertex $v^\frown(i)$ for every $i < 2$ and $v \in L^c_n$. Adding the path

$$((e^n_1)^\frown(0), p_0, p_1, \ldots, p_{c(n)}, (e^n_1)^\frown(1))$$

completes the construction of L^c_{n+1}, and its endpoints are

$$e_i^{n+1} = (e_0^n)^\frown(i) = (p_0)^\frown(0)^{n\frown}(i),$$

for $i < 2$. This finishes the recursion. Note that to construct L^c_{n+1} from L^c_n, one only requires knowledge of the value $c(n)$.

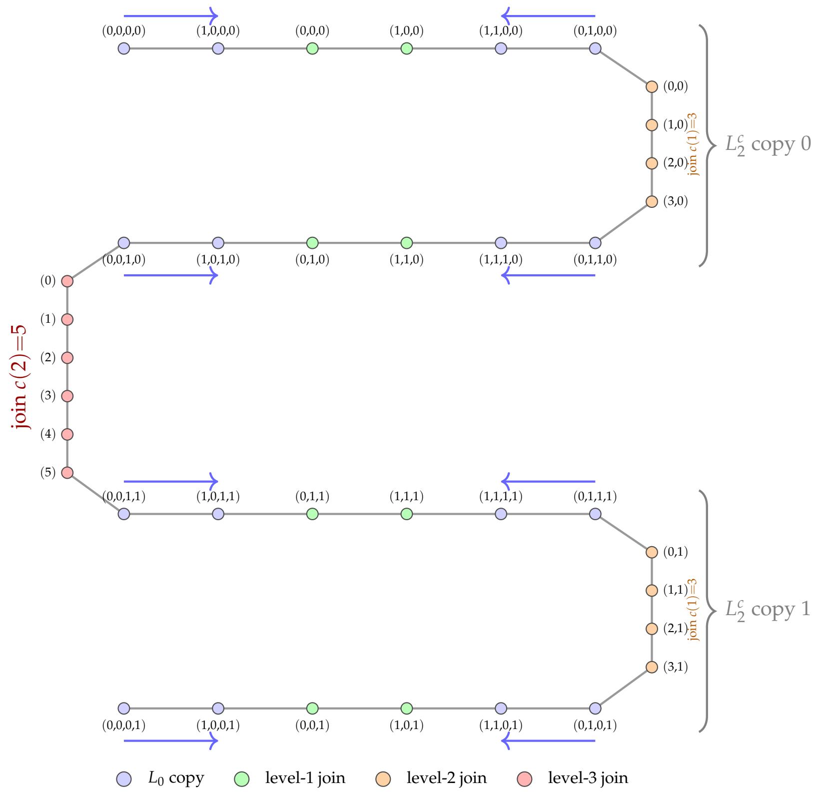

Figure 1: L^c_3 for $c(0) = 1, c(1) = 3, c(2) = 5$.

To develop some intuition on what (1) says about a vertex v in L^c_n: the number $|v| - 1$ indicates that v was added that many stages ago in the recursion, at the position of p_k in the path joining the two copies of the previous paths; the sequence t tells the story, in order, of which of the two copies of the previous path the vertex lies in. In other words, vertices of the form $(p_k)^\frown t$ for $t \in 2^n$ are precisely those that were copied from the original L^c_0 in all possible positions along L^c_n, doubling the number of copies at each stage.

Based on this discussion, a vertex in some L^c_n is completely determined by the stage $n - m$ at which it was added in the position of p_k and then copied according to the binary sequence t.

3 A notion of smallness for copies

Now, define a notion of smallness analogous to the one presented in [Ber24]. By definition, the set of edges of G is the continuous image of some Polish space E, let us say under π. For a finite graph H, consider $\mathrm{Hom}(H, G)$ as the set of all maps

$$\varphi\colon V(H) \to X$$
$$\varphi\colon E(H) \to E$$

that satisfy that for all $uv \in E(H)$, $\pi(\varphi(uv)) = \varphi(u)\varphi(v)$.[1] Recall that $X = V(G)$ is the Polish space of vertices on which the analytic graph G is defined. Notice that $\mathrm{Hom}(H, G)$ is also Polish space, being a closed subset of the product space $X^{V(H)} \times E^{E(H)}$.

Given $\mathcal{H} \subseteq \mathrm{Hom}(H, G)$, define $\mathcal{H}(u) := \{\varphi(u) : \varphi \in \mathcal{H}\}$, which is analytic whenever $\mathcal{H}$ is Borel. Also, let $\mathcal{H}(uv) := \{\varphi(uv) : \varphi \in \mathcal{H}\}$.

Definition 3.1.

1. Let $k < \omega$. Say that the set A has property $\Phi(A, k)$ if $A \subseteq X$ is an analytic set of vertices of G such that all odd walks of G with endpoints in A have length at most $2k - 1$.

2. A Borel set $\mathcal{L} \subseteq \mathrm{Hom}(L^c_n, G)$ is called *tiny* if there is a vertex $u \in V(L^c_n)$ and a natural k such that $\Phi(\mathcal{L}(u), k)$.

3. A set $\mathcal{L} \subseteq \mathrm{Hom}(L^c_n, G)$ is *small* if it is the countable union of tiny sets. This notion defines a σ-ideal.

4. A set is *large* if it is not small. Notice that when H is the graph with a single vertex $H = \bullet\, (= L^c_0)$, $\mathrm{Hom}(H, G)$ can be identified with X and then for any $\mathcal{H} \subseteq \mathrm{Hom}(\bullet, G)$, $\mathcal{H}(\bullet)$ is identified with $\mathcal{H}$.

Now invoke the following auxiliary result that links the Φ property to Borel 2-colourability. The proof proceeds by induction on n using analytic separation; see [CMSV21, Claims 3.3–3.5] for details. Recall that a set of vertices of G is *invariant* if it is the union of connected components of G. In addition, if $A \subseteq X$, then $[A]_{E_G}$ is the union of all connected components of G that intersect A, that is, the set of vertices that can be reached from some element of A by a finite path. Therefore, a set A is invariant if and only if $A = [A]_{E_G}$.

Lemma 3.2 (Claim 3.5 of [CMSV21])**.** *If $\Phi(A, n)$, then there is an invariant Borel set $B \supseteq [A]_{E_G}$ that induces a 2-Borel-colourable subgraph of G, that is, there is a Borel proper colouring $c_B\colon G\restriction_B \to 2$.*

[1] Formally, φ is a pair of two maps, however one can assume that $V(H) \cap E(H) = \emptyset$ and then there is no risk of confusion.

Proposition 3.3. *If $\chi_B(G) > 2$ then X (via its identification with* $\mathrm{Hom}(\bullet, G)$*) is large.*

Proof. Suppose that $\mathrm{Hom}(\bullet, G) = \bigcup_{m<\omega} \mathcal{H}_m$ where each $\mathcal{H}_m$ is a tiny Borel set. For every $m < \omega$, Lemma 3.2 produces an invariant Borel set $B_m \supseteq [\mathcal{H}_m]_{E_G}$ with a Borel 2-colouring c_m. Define $c\colon X \to 2$ by letting $m(x) := \min\{m < \omega : x \in B_m\}$ and $c(x) = c_{m(x)}(x)$. Then c is a Borel 2-colouring because each B_m is invariant. □

Given that L^c_{n+1} is obtained from two copies of L^c_n joined by a path, it is convenient to have a means to refering to these distinct copies. Keeping property (1) in mind, for every $\varphi \in \mathrm{Hom}(L^c_{n+1}, G)$ and $i < 2$, define $\varphi^i \in \mathrm{Hom}(L^c_n, G)$ by $\varphi^i(u) = \varphi(u^\frown(i))$ for all $u \in V(L^c_n)$ and $\varphi^i(uv) = \varphi(u^\frown(i), v^\frown(i))$ for all $(u,v) \in E(L^c_n)$. Thus, given $\mathcal{L} \subseteq \mathrm{Hom}(L^c_n, G)$, define $\mathcal{L}^{+c}$ as the set of all $\varphi \in \mathrm{Hom}(L^c_{n+1}, G)$ for which $\varphi^i \in \mathcal{L}$ for both $i < 2$. Notice that if $\mathcal{L}$ is Borel, then so is $\mathcal{L}^{+c}$.

The length of the path added needs to be updated dynamically throughout the proof. Formally, one adjusts the parameter c by moving into a different branch of the Baire space when constructing the graphs L^c_n. Notice that, for two $c, d \in \omega^\omega$ and for every $n < \omega$, the condition $d\restriction_n = c\restriction_n$ implies that $L^c_k = L^d_k$ for all $k < n$.

Lemma 3.4.

(a) *Suppose that $n > 0$, that c only takes odd values, that k is a natural and $t \in 2^{<n}$ is such that $(p_k)^\frown t^\frown(i)$, for $i < 2$, are vertices of L^c_n (see (1)). Then, they are an odd distance apart in L^c_n.*

(b) *For any $n < \omega$, if $\mathcal{L} \subseteq \mathrm{Hom}(L^c_n, G)$ is a large Borel set, then for any $N < \omega$ there exists $d \in \omega^\omega$ such that $d\restriction_n = c\restriction_n$, $d(n) \geq N$ is odd, and $\mathcal{L}^{+d}$ is a nonempty subset of* $\mathrm{Hom}(L^d_{n+1}, G)$*.*

(c) *For any $n < \omega$, if $\mathcal{L} \subseteq \mathrm{Hom}(L^c_n, G)$ is a large Borel set, then there exists $d \in \omega^\omega$ such that $d\restriction_n = c\restriction_n$ and $\mathcal{L}^{+d}$ is a large subset of* $\mathrm{Hom}(L^d_{n+1}, G)$*.*

Proof. For (a), observe that both vertices come from the same vertex $(p_k)^\frown t$ in $L^c_{|t|+1}$, and so their distance in L^c_n is twice the distance from $(p_k)^\frown t$ to $e^{|t|}_1$ in $L^c_{|t|+1}$ (since the two vertices are each in distinct copies of $L^c_{|t|+1}$ inside L^c_n and the unique path between them passes through the joining path at stage $|t|$), plus the length of the joining path at that stage, which is $c(|t|) + 2$. Since $c(|t|)$ is odd, this distance is odd.

For (b), since $\mathcal{L}$ is large, it is not tiny. Hence, for every vertex $u \in V(L^c_n)$ and every natural k, $\Phi(\mathcal{L}(u), k)$ fails. In particular, fix the gluing vertex $u_0 := e^n_1 \in V(L^c_n)$ (recall that $e^0_1 = p_0$). Then for every k there exist $\varphi_0, \varphi_1 \in \mathcal{L}$ and an odd walk $(v_0, v_1, \dots, v_m)$ in G with $v_0 = \varphi_0(u_0)$, $v_m = \varphi_1(u_0)$, and $m > 2k - 1$. Choosing k large enough, it is ensured that $m \geq N + 2$ and m is odd. Since m is odd, $d(n) := m - 2$ is also odd and satisfies $d(n) \geq N$. For each $1 \leq j \leq m$, since $(v_{j-1}, v_j) \in E(G) = \pi[E]$, pick $e_j \in E$ with $\pi(e_j) = (v_{j-1}, v_j)$. Setting $d(x) = c(x)$ for all $x \neq n$ and noting $L^c_n = L^d_n$ (since $d\restriction_n = c\restriction_n$), define $\varphi \in \mathrm{Hom}(L^d_{n+1}, G)$ by $\varphi^0 = \varphi_0$, $\varphi^1 = \varphi_1$, $\varphi(p_i) = v_{i+1}$ for $0 \leq i \leq m - 2 = d(n)$, and the edge assignments $\varphi(u_0^\frown(0), p_0) = e_1$, $\varphi(p_j, p_{j+1}) = e_{j+1}$ for $0 \le j < m-2$, and $\varphi(p_{m-2}, u_0^\frown(1)) = e_m$. Then $\varphi \in \mathcal{L}^{+d}$.

Next, prove (c) by contradiction. Recall that constructing L^c_{n+1} from L^c_n only requires knowing the value of $c(n)$. Suppose that for all d that agree with c except possibly in the n-th value, $\mathcal{L}^{+d} = \bigcup_m \mathcal{F}^d_m$ where each $\mathcal{F}^d_m$ is a tiny Borel subset of $\mathrm{Hom}(L^d_{n+1}, G)$. For each m and d, let $w^d_m \in V(L^d_{n+1})$ and $k^d_m < \omega$ be such that $\Phi(\mathcal{F}^d_m(w^d_m), k^d_m)$.

Since $w^d_m \in V(L^d_{n+1})$, it is either a *non-path vertex* of the form $u^\frown(i)$ for some $u \in V(L^c_n)$ and $i < 2$, or a *path vertex* p_j for some $0 \le j \le d(n)$ in the joining path. We claim that in either case, there exists a vertex $u^d_m \in V(L^c_n)$ and $\ell^d_m < \omega$ such that $\Phi(\mathcal{F}^d_m({u^d_m}^\frown(i^d_m)), \ell^d_m)$ for some $i^d_m < 2$ where ${u^d_m}^\frown(i^d_m)$ is a non-path vertex of L^d_{n+1}.

Indeed, if $w^d_m = u^\frown(i)$ is already non-path, take $u^d_m = u$, $i^d_m = i$, $\ell^d_m = k^d_m$. If $w^d_m = p_j$ is a path vertex of L^d_{n+1}, then the joining path connects ${e^n_1}^\frown(0)$ to ${e^n_1}^\frown(1)$ via $p_0, \dots, p_{d(n)}$. Since p_j and ${e^n_1}^\frown(0)$ are at distance $j+1$ in L^d_{n+1}, every homomorphism $\varphi \in \mathcal{F}^d_m$ maps them to vertices connected by a walk of lengh $j+1$ in G. Given any odd walk $(a_0, \dots, a_r)$ in G with $a_0, a_r \in \mathcal{F}^d_m({e^n_1}^\frown(0))$, witnessed by $\varphi, \varphi' \in \mathcal{F}^d_m$, the walk $(\varphi(p_j), \dots, \varphi({e^n_1}^\frown(0)) = a_0, \dots, a_r = \varphi'({e^n_1}^\frown(0)), \dots, \varphi'(p_j))$ has length $r + 2(j+1)$, which has the same parity as r (hence is also odd), and has endpoints in $\mathcal{F}^d_m(p_j)$. By $\Phi(\mathcal{F}^d_m(p_j), k^d_m)$, $r + 2(j+1) \le 2k^d_m - 1$, so $r \le 2(k^d_m - j - 1) - 1$. It follows that $\Phi(\mathcal{F}^d_m({e^n_1}^\frown(0)), \max\{0, k^d_m - j - 1\})$. Thus, set $u^d_m = e^n_1$, $i^d_m = 0$, $\ell^d_m = \max\{0, k^d_m - j - 1\}$.

Claim 3.4.1. *For each m and d, there is a Borel set $B^d_m \supseteq \mathcal{F}^d_m({u^d_m}^\frown(i^d_m))$ such that $\Phi(B^d_m, \ell^d_m)$.*

Proof of Claim. Use the First Reflection Theorem (Lemma 1.4). Fix $\ell = \ell^d_m$. It suffices to verify that $\Phi_\ell := \{A \subseteq X : \Phi(A, \ell)\}$ is Π^1_1 on Σ^1_1. Let Y be Polish and $A \subseteq Y \times X$ be Σ^1_1. Fix a Polish space N and a continuous surjection $g \colon N \to A$. For each odd $r \ge 2\ell + 1$, define

$$\begin{aligned} C_r := \{&(y, x_0, \dots, x_r, n_0, n_1, \hat{e}_1, \dots, \hat{e}_r) \in Y \times X^{r+1} \times N^2 \times E^r : \\ &g(n_0) = (y, x_0) \wedge g(n_1) = (y, x_r) \wedge \textstyle\bigwedge_{j<r} \pi(\hat{e}_{j+1}) = (x_j, x_{j+1})\}. \end{aligned}$$

Then C_r is closed (by the continuity of g and π), and $D_r := \mathrm{proj}_Y(C_r)$ is Σ^1_1. Hence $D := \bigcup\{D_r : r \text{ odd}, r \ge 2\ell + 1\}$ is Σ^1_1, and $A_{\Phi_\ell} = \{y \in Y : A_y \in \Phi_\ell\} = Y \setminus D$ is Π^1_1. Since $\mathcal{F}^d_m({u^d_m}^\frown(i^d_m))$ is analytic and satisfies $\Phi(\cdot, \ell)$, the First Reflection Theorem yields the desired Borel. $\dashv$

Take

$$\mathcal{H}^d_m = \{\varphi \in \mathcal{L} : \varphi(u^d_m) \in B^d_m\},$$

which is also a Borel subset of $\mathrm{Hom}(L^c_n, G)$. We claim that $\mathcal{H}^d_m$ is tiny. Indeed, $\mathcal{H}^d_m(u^d_m) = \{\varphi(u^d_m) : \varphi \in \mathcal{L} \wedge \varphi(u^d_m) \in B^d_m\} \subseteq B^d_m$, so $\Phi(\mathcal{H}^d_m(u^d_m), \ell^d_m)$.

Now, the set of pairs (u^d_m, i^d_m) ranges over the finite set $V(L^c_n) \times 2$ (which does not depend on d), and (m, d) range over countably many values. Thus $\bigcup_{m,d} \mathcal{H}^d_m$ is a countable union of tiny sets, hence small. The set $\mathcal{L}_- := \mathcal{L} \setminus \bigcup_{m,d} \mathcal{H}_m^d$ is therefore large (and Borel). By part (b), there exist φ and d such that $\varphi \in \mathcal{L}_-^{+d} \subseteq \mathcal{L}^{+d}$. Then $\varphi \in \mathcal{F}_m^d$ for some m. It follows that $\varphi^{i_m^d}(u_m^d) = \varphi(u_m^d {}^\frown (i_m^d)) \in \mathcal{F}_m^d(u_m^d {}^\frown (i_m^d)) \subseteq B_m^d$, hence $\varphi^{i_m^d} \in \mathcal{H}_m^d$. But $\varphi \in \mathcal{L}_-^{+d}$ implies $\varphi^{i_m^d} \in \mathcal{L}_-$, contradicting $\mathcal{L}_- \cap \mathcal{H}_m^d = \emptyset$. □

4 Construction of a minimal graph of Borel chromatic number at least 3

The goal of this section is to construct a family of graphs $\mathbb{L}_c$ indexed by $c \in \omega^\omega$.

Fix c and consider X_c as the set of all tuples $(m, k, x) \in \mathbb{N} \times \mathbb{N} \times 2^{\mathbb{N}}$ such that either $m = 0$ and $k = 0$, or $m \geq 1$ and $k \leq c(m-1)$; this is a closed subspace of the product space, and hence Polish. Define, for each (m, k, x) with $m \leq n$, $\pi_n(m, k, x)$ as the vertex of L_n^c determined by these parameters. Formally, $\pi_n(m, k, x) := (p_k)^\frown x \restriction_{n-m}$. For example, $\pi_3(1, 2, 0110 \cdots) = (p_2, 0)$ can be seen in Figure 1 labeled as $(2, 0)$.

Finally, $\mathbb{L}_c$ is the graph on X_c where two (n_i, k_i, x_i), for $i < 2$, are adjacent if and only if the $\pi_n(n_i, k_i, x_i)$ are adjacent in all L_n^c with $n \geq \max\{n_0, n_1\}$. Some basic properties of $\mathbb{L}_c$ follow.

Lemma 4.1.

(a) Two vertices $(n_i, k_i, x_i) \in X_c$, for $i < 2$, are in the same connected component of $\mathbb{L}_c$ if and only if there are $t_i \in 2^{<\omega}$ and $x \in 2^\omega$ such that $|t_0| - |t_1| = n_1 - n_0$ and $x_i = t_i^\frown x$.

(b) $\mathbb{L}_c$ has no cycles and all vertices of $\mathbb{L}_c$ have degree 2 except for the vertex $(0, 0, \{0\}^\omega)$, which has degree 1.

(c) If, additionally, c takes only odd values, then $\chi_B(\mathbb{L}_c) = 3$.

Proof. For (a), let $(n_i, k_i, x_i) \in X_c$ for $i < 2$. If they are in the same connected component, then for all large enough n, $\pi_n(n_0, k_0, x_0)$ and $\pi_n(n_1, k_1, x_1)$ lie in the same copy of L_{n-1}^c inside L_n^c. By the construction of L_n^c, two vertices $(p_{k_0})^\frown x_0 \restriction_{n-n_0}$ and $(p_{k_1})^\frown x_1 \restriction_{n-n_1}$ are in the same copy of L_{n-1}^c in L_n^c if and only if their last binary digits agree, that is, $x_0(n - n_0 - 1) = x_1(n - n_1 - 1)$ for all large n. Writing $x_0 = t_0^\frown x$ and $x_1 = t_1^\frown x$ with $|t_0| = n_1 - n_0 + |t_1|$ (assuming $n_0 \leq n_1$), one obtains the stated condition. Conversely, if $x_i = t_i^\frown x$ with $|t_0| - |t_1| = n_1 - n_0$, then for all large n, $\pi_n(n_0, k_0, x_0) = (p_{k_0})^\frown t_0^\frown x \restriction_r$ and $\pi_n(n_1, k_1, x_1) = (p_{k_1})^\frown t_1^\frown x \restriction_r$ for some r, so they share a common tail and lie in the same connected component of L_n^c, hence of $\mathbb{L}_c$.

For (b), every vertex of L_n^c has degree at most 2, so the same holds for $\mathbb{L}_c$. It follows from acyclicity of the finite paths L_n^c that $\mathbb{L}_c$ is acyclic. The vertex $(0, 0, 0^\omega)$ maps under π_n to $e_0^n = (p_0)^\frown 0^n$, which is an endpoint of L_n^c, hence has degree 1 in L_n^c for all $n \geq 1$. Thus

$(0, 0, 0^\omega)$ has degree 1 in $\mathbb{L}_c$. For any other vertex $(m, k, x) \in X_c$, the image $\pi_n(m, k, x)$ is an interior vertex of L^c_n for all sufficiently large n, so it has degree exactly 2.

(c) That $\chi_B(\mathbb{L}_c) \leq 3$ follows from Proposition 1.2 and part (b). Argue the other inequality by contradiction. Suppose that there is a Borel partition of X_c into two independent sets $X_c = I_0 \cup I_1$. Since X_c is Polish, the Baire category theorem guarantees that some I_ε is non-meager, hence comeager in a basic open set $U = \{(m, k, x) \in X_c : x{\restriction}_r = t\}$ for some m, k, r, and $t \in 2^r$. In particular, for a comeager set of $x \in 2^{\mathbb{N}}$ extending t, $(m, k, x) \in I_\varepsilon$. Since $(m, k, t^\frown(0)^\frown y)$ and $(m, k, t^\frown(1)^\frown y)$ are in the same connected component (by part (a)) for any $y \in 2^{\mathbb{N}}$, and I_ε is comeager in U, one can find y such that both $(m, k, t^\frown(0)^\frown y)$ and $(m, k, t^\frown(1)^\frown y)$ lie in I_ε. But by Lemma 3.4(a), these two vertices are an odd distance apart in $\mathbb{L}_c$ (as $(p_k)^\frown t^\frown(0)$ and $(p_k)^\frown t^\frown(1)$ are an odd distance apart in every L^c_n for large enough n), contradicting I_ε being independent. □

The main result is then split into two parts.

Theorem 4.2. *For any analytic graph G, either $\chi_B(G) \leq 2$ or there is a c such that $\mathbb{L}_c \to_c G$.*

Moreover, c can be chosen to be unbounded and take only odd values.

Theorem 4.3. *Let $c, d \in \omega^\omega$ be unbounded and only taking odd values, then $\mathbb{L}_c$ and $\mathbb{L}_d$ are continuously homomorphically equivalent.*

The proof of Theorem 4.3 is purely combinatorial: one constructs homomorphisms between $\mathbb{L}_c$ and $\mathbb{L}_d$ by iteratively extending partial maps along finite path components, using the "large gap property" guaranteed by unboundedness. The argument does not depend on the method used to establish Theorem 4.2; see [CMSV21, Proposition 4.2 and Claim 4.1] for details.

Combining Theorems 4.2 and 4.3, and fixing any unbounded odd c_0 (e.g., $c_0(0) = 1$ and $c_0(n) = 2n - 1$ for $n > 0$)[2], the L_0 dichotomy of [CMSV21] is recovered:

Corollary 4.4. *Let $\mathbb{L}_0 := \mathbb{L}_{c_0}$. For any analytic graph G on a Polish space, exactly one of the following holds:*

(a) $\chi_B(G) \leq 2$;

(b) $\mathbb{L}_0 \to_c G$.

5 Proof of Theorem 4.2

By Lemma 4.1(c) and Fact 1.1, it is clear that both conditions cannot hold simultaneously. Now suppose that G is an analytic graph of Borel chromatic number at least three. The theorem is proved by constructing a valid c and a continuous homomorphism witnessing $\mathbb{L}_c \to_c G$.

[2] It is perhaps worth noting that I picked a *different* parameter for Figure 1 purely for aesthetic purposes.

By Proposition 3.3, X is large. Fix compatible complete metrics ρ on X and δ on E. Since $L_0^c = \bullet$, $\mathrm{Hom}(L_0^c, G)$ may be identified with X and set $\mathcal{L}_0 := X$, which is large. Recursively construct a function $c \in \omega^\omega$ (taking only odd values) and a sequence of large Borel sets $\mathcal{L}_n \subseteq \mathrm{Hom}(L_n^c, G)$ such that for all n,

(1) $\mathcal{L}_{n+1} \subseteq \mathcal{L}_n^{+c}$;

(2) for all $u \in V(L_n^c)$, $\mathrm{diam}_\rho(\overline{\mathcal{L}_n(u)}) < 2^{-n}$; and

(3) for all $(u, v) \in E(L_n^c)$, $\mathrm{diam}_\delta(\overline{\mathcal{L}_n(u, v)}) < 2^{-n}$.

Given $\mathcal{L}_n$ large, apply Lemma 3.4(c) to obtain $d \in \omega^\omega$ with $d\restriction_n = c\restriction_n$ and $\mathcal{L}_n^{+d}$ large; set $c(n) := d(n)$. To arrange conditions (2) and (3) for $n+1$: since X and E can each be covered by countably many closed sets of ρ-diameter (respectively δ-diameter) less than $2^{-(n+1)}$, and small sets form a σ-ideal, one may intersect with the preimages of these covers and discard the resulting small pieces, passing to a large Borel subset $\mathcal{L}_{n+1} \subseteq \mathcal{L}_n^{+c}$ satisfying (2) and (3) at stage $n+1$. Moreover, by choosing $c(n)$ sufficiently large using part (b) at each stage, c can be made unbounded.

By (1), for all $n \geq m$,

$$\mathcal{L}_{n+1}(\pi_{n+1}(m, k, x)) \subseteq \mathcal{L}_n(\pi_n(m, k, x));$$

indeed, given that $\pi_{n+1}(m, k, x) = \pi_n(m, k, x)^\frown(x(n-m))$, each $\varphi \in \mathcal{L}_{n+1}$ satisfies $\varphi^{x(n-m)} \in \mathcal{L}_n$ and

$$\varphi^{x(n-m)}(\pi_n(m, k, x)) = \varphi(\pi_{n+1}(m, k, x)).$$

By the same reasoning, if $\pi_n(m_0, k_0, x_0)$ and $\pi_n(m_1, k_1, x_1)$ are adjacent in L_n^c for all large n, then

$$\begin{aligned} &\mathcal{L}_{n+1}(\pi_{n+1}(m_0, k_0, x_0), \pi_{n+1}(m_1, k_1, x_1)) \\ &\quad \subseteq \mathcal{L}_n(\pi_n(m_0, k_0, x_0), \pi_n(m_1, k_1, x_1)). \end{aligned}$$

Thus, the sets $\overline{\mathcal{L}_n(\pi_n(m, k, x))}$ are closed and nested, with diameters tending to zero (see (2) from before). Since ρ is a complete metric, Cantor's intersection theorem applies. For $(m, k, x) \in X_c$, let $f(m, k, x)$ be the unique point in

$$\bigcap_{n=m}^{\infty} \overline{\mathcal{L}_n(\pi_n(m, k, x))} = \bigcap_{n=0}^{\infty} \overline{\mathcal{L}_{n+m}((p_k)^\frown x\restriction_n)}.$$

The map $f \colon X_c \to X$ is continuous.

It remains to check that it is a homomorphism from $\mathbb{L}_c$ to G. Suppose that $(m_i, k_i, x_i)_{i<2}$ are adjacent in $\mathbb{L}_c$, that is, for all $n \geq n_0 := \max\{m_0, m_1\}$, $\pi_n(m_i, k_i, x_i)$ are adjacent in L_n^c.

Similar to the preceding paragraph, the fact that δ is a complete metric implies that there is a unique edge $e \in E$ inside

$$\bigcap_{n \geq n_0} \overline{\mathcal{L}_n(\pi_n(m_0, k_0, x_0), \pi_n(m_1, k_1, x_1))}.$$

By the continuity of π, $(f(m_0, k_0, x_0), f(m_1, k_1, x_1)) = \pi(e) \in E(G)$, in other words, the images form an edge of G, and this concludes the proof.

The hereby presented argument shows that the core of the L_0 dichotomy, namely the construction of a continuous homomorphism from an $\mathbb{L}_c$ graph into any analytic graph of Borel chromatic number at least three, admits a proof that closely parallels Bernshteyn's argument for the G_0 dichotomy, with the challenge of adding a dynamically-changing parameter to the construction.

The main new ingredient is the replacement of the independence-based σ (sets whose vertex projections are independent) with the odd-walk-bounded σ-ideal defined by the Φ property. It remains an interesting question whether analogous adaptations can yield simplified proofs of other dichotomy results in Borel combinatorics, or even entirely new ones.

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