Infinitary Amoebas

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Abstract

Here goes the abstract.

1 Introduction

There are several reasons why studying countable graphs is desirable, but they can all be somewhat summarized by the fact that if X is countable, X^X is a separable, completely metrizable topological space. Two particularly interesting issues arise in the case when X is uncountable, which I present in Section 3.

2 Preliminaries

3 Uncountable graphs

Automorphisms, and by extension, feasible edge-replacements, need not be approximable by a sequence of automorphisms in a given set, even when said automorphism is topologically close.

Proposition 3.1. *Let G be the graph on the vertex set 2^ω with edges (x, y) where $x{\restriction}_{\omega\setminus 1} \neq y{\restriction}_{\omega\setminus 1}$.*

Consider the map $s\colon 2^\omega \to 2^\omega$ that flips the first bit of an infinite binary sequence, that is, for every $x \in 2^\omega$, $s(i^\frown x) := (1-i)^\frown x$. For each finite set $F \subseteq 2^\omega$, define

$$\sigma_F(x) := \begin{cases} s(x), & x \in F; \\ x, & x \notin F. \end{cases}$$

It is straightforward to show that $\{\sigma_F : F \in [2^\omega]^{<\omega}\} \cup \{s\} \subseteq A(G) = \mathrm{Fer}(G)$.

4 Countable graphs

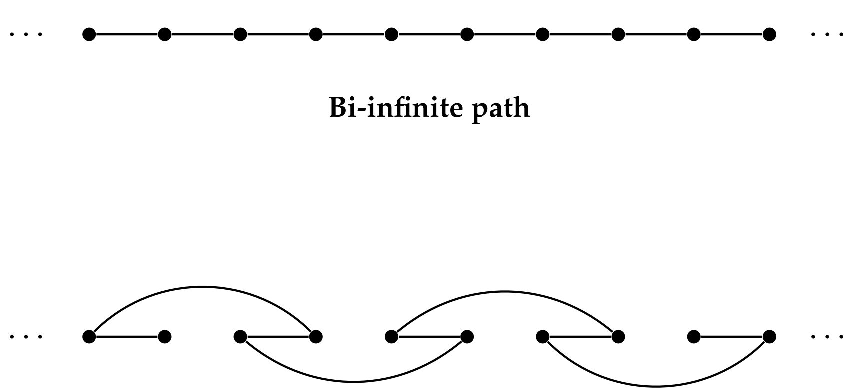
For this section, assume that G is a graph defined on an infinite countable set X. In this case, X^X is homeomorphic with the Baire space ω^ω, and so we are dealing with separable and completely metrizable spaces (i.e. Polish spaces). One can show that S_X, being a G_δ subspace of the Baire space, is also Polish and thus its topology is induced by a complete metric. Indeed, observe that

$$S_X = \bigcap_{\substack{x,y\in X\\ x\neq y}} \{f \in X^X : f(n) \neq f(m)\} \cap \bigcap_{y\in X} \bigcup_{x\in X} \{f \in X^X : f(x) = y\}$$

is a countable (whenever X is) intersection of open sets. Giving an explicit complete metric is done in the example at the end of the section.

Group actions on graphs may be quite different in the infinite case. For instance, consider the automorphism of a random graph. Missing references. Erdos, Renyi, Rado, etc. Most finite random graphs are asymmetric but almost all countably infinite random graphs are highly symmetric.

Many properties of finite amoebas translate to our more general definition, such as the fact that $\text{Fer}(G) = \text{Fer}(\overline{G})$ for any graph G. Also, any complete graph on any set X is a local amoeba, and thus so is the discrete graph (the graph with no edges). As a first non-trivial example, the bi-infinite path $P_\mathbb{Z}$, i.e. the graph defined on $X = \mathbb{Z}$ where the edges are given by consecutive pairs, is interestingly not a local amoeba. In fact in this context, bi-infinite paths play a similar role as finite cycles. Indeed, it straightforward to verify that $\text{Fer}(P_\mathbb{Z}) = \text{Aut}(P_\mathbb{Z})$. Notice that the basic open set $[(0\,1)(2)]$ is disjoint from $\text{Fer}(G)$ since the only automorphism interchanging 0 and 1 is a reflection (which doesn't fix the element 2), and thus $(0\,1) \notin \overline{\text{Fer}(G)}$.



Bi-infinite path

Bi-infinite twisted path

Figure 1: Infinite paths.

Lemma 4.1. *Let $\Gamma \leq S_X$ and $\varphi \in S_X$. Then $\varphi \in \overline{\Gamma}$ if and only if for every finite $F \subseteq X$, there is $\sigma \in \Gamma$ such that $\varphi\restriction_F = \sigma\restriction_F$.*

Proof. For the forward direction, take φ and F as above. Clearly, $[\varphi\restriction_F]$ is a basic open neighborhood of φ and hence must intersect Γ. Any σ witnessing this has the sought property.

Conversely, take an arbitrary $t \in \omega^{<\omega}$ extended by φ. Pick $\sigma \in \Gamma$ with $\varphi\restriction_{\operatorname{dom} t} = \sigma\restriction_{\operatorname{dom} t}$. Clearly, $\sigma \in \Gamma \cap [t]$, proving that $\varphi \in \overline{\Gamma}$. □

A group action on X is called *n-transitive* if $|X| \geq n$ and for any two pairwise distinct n-tuples $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$, there is a group element g such that $gx_i = y_i$ for all $1 \leq i \leq n$. Notice that 1-transitive is just the usual definition of *transitive*, i.e. that there is a single orbit. If for all $n \geq 1$, $\operatorname{Fer}(G)$ acts n-transitively on X, we say $\operatorname{Fer}(G)$ is *highly transitive*.

Proposition 4.2. *The following are equivalent.*

(1) The graph G is a local amoeba.

(2) Every finite $F \subseteq X$ and every $\varphi \in S_X$, there is $\sigma \in \operatorname{Fer}(G)$ such that $\varphi\restriction_F = \sigma\restriction_F$.

(3) $\operatorname{Fer}(G)$ is highly transitive.

Proof. The implication [$1 \implies 2$] follows from the previous lemma. Assume (2) and let $n \geq 1$ and $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ be two pairwise distinct. Let $F := \{x_i : 1 \leq i \leq n\}$, then fix a bijection $\omega \setminus F \to \omega \setminus \{y_i : 1 \leq i \leq n\}$ and define φ such that it agrees with said bijection and in addition satisfies $\varphi(x_i) = y_i$. By (2), some $\sigma \in \operatorname{Fer}(G)$ has the desired property.

Finally, assume (3), let φ be an arbitrary permutation on X and $F = \{x_1, \ldots, x_n\} \subseteq X$. Apply (3) to the tuples $(x_1, \ldots, x_n)$ and $(\varphi(x_1), \ldots, \varphi(x_n))$ to get $\sigma \in \operatorname{Fer}(G)$. It follows that σ agrees with φ on F. Since F was arbitrary, by the previous lemma, G must be a local amoeba. □

I point out that in the example of the bi-infinite path, $\operatorname{Fer}(P_{\mathbb{Z}})$ acts transitively on $P_{\mathbb{Z}}$ but not 2-transitively, as the same aforementioned example clearly shows, namely the pairs $(0, 2)$ and $(1, 2)$. In such cases, one could use the largest n for which the group acts n-transitively on G to compare how close G is to being a local amoeba. Work in progress. Is there a graph with an n-transitive Fer group that is not $(n+1)$-transitive? This is likely an open problem: see Introduction https://arxiv.org/pdf/1501.04182

Denote by P_{n+1} the path on the vertex set $n+1$ with edges given by consecutive pairs of numbers. For a label $i \in X$, denote by $\operatorname{Fer}^i(G)$ the subgroup of $\operatorname{Fer}(G)$ generated by those generators of $\operatorname{Fer}(G)$ that fix i. We say a graph G is *stem-symmetric with respect to i* (or to the vertex labeled by i) if $\operatorname{Fer}^i(G)$ is the symmetric group on $X \setminus \{i\}$.

The next result provides an example of a graph where the gap between the automorphism group and the Fer group is as large as possible. Concretely, the countably infinite path on $X = \omega$, denoted P_ω where the edges are given by consecutive pairs of naturals, has a trivial automorphism group, but $\mathrm{Fer}(P_\omega)$ is dense in S_ω. This comment applies to a different graph, maybe the Rado graph. In particular, this illustrates that infinite amoebas are far richer than their finite counterparts, where the only finite connected local amoebas G satisfying $\mathrm{Fer}(G) = \mathrm{Aut}(G)$ are the complete graphs (see Proposition 2.4 of [**?**]).

We need the following combinatorial lemma to prove that infinite paths are local amoebas. The only need the fact that finite paths are stem-symmetric with respect to an endpoint.

Lemma 4.3. *The graph P_{n+1} is stem-symmetric with respect to k if and only if $k \le n$ and, if $n \ge 5$ is odd, $k \ne \frac{n-1}{2}$.*

Proof. We prove the case $k = n$ as the others are similar. Let $1 \le \ell < n$ and notice that the graph $P_{n+1} - (\ell, \ell+1) + (0, \ell+1)$ is a path on $n+1$ vertices and thus isomorphic to P_{n+1}. We can find an explicit isomorphism that fixes n and decompose it into transpositions. In symbols,

$$\prod_{k=0}^{\ell}(k \quad \ell - k) : (\ell\ (\ell+1) \to 0\ (\ell+1)).$$

The second step is realizing that these permutations indeed generate the symmetric group on (the set) n. This is straightforward and can be done in many ways. For example, every transposition of the form $(0\ \ell)$, for $1 \le \ell < n$, is in $\mathrm{Fer}^n(P_{n+1})$ by an inductive argument on ℓ, taking conjugates and noticing that the transposition in question is the left-most factor in the ℓ-th permutation above. □

Proposition 4.4. *P_ω is a local amoeba.*

Proof. Let $F \subseteq \omega$ be finite and $\varphi \in S_\omega$. Define $n := \max(F \cup \varphi[F]) + 1$ and notice that $t := \varphi\restriction_F$ is a bijection between finite subsets of n. In particular, there exists a bijection $s : n \setminus F \to n \setminus t[F]$. Now define $\sigma : X \to X$ by

$$\sigma(x) := \begin{cases} s(x) & x \in n \setminus F \\ t(x) & x \in F \\ x & x \ge n \end{cases}$$

Routine arguments show that $\sigma \in S_X$. Now consider the induced subgraph P_{n+1} and the permutation $\sigma\restriction_{n+1}$. Observe that this permutation fixes the endpoint of the path, n, by definition. Thus by the previous lemma, there is a sequence $f_0, \ldots, f_m$ of generators of $\mathrm{Fer}(P_{n+1})$, all of which fix n, such that $\sigma\restriction_{n+1} = f_0 \cdots f_m$.

For each $i < m$, we can extend f_i to a permutation $\overline{f_i}$ on X by fixing every label $x > n$. Clearly, each $\overline{f_i}$ is a member of $\mathrm{Fer}(P_\omega)$ and thus $\sigma = \overline{f_0} \cdots \overline{f_m} \in \mathrm{Fer}(P_\omega)$. Finally, the choice of σ makes it evident that $\sigma\restriction_F = t = \varphi\restriction_F$, and thus P_ω is a local amoeba by Proposition 4.2. □

Here is a generalization of all of these easy examples:

Theorem 4.5. *Suppose that G is a graph on $X = \omega$ such that for any finite $F \subseteq X$, there is a finite induced subgraph $H \subseteq G$ whose vertices contain F Let's do other stuff before this so we understand it better.*

To emphasize the value of the discussed results, we compute an explicit example that the author thinks is not trivial to deduce without the topological tools developed in this section.

Consider the graph A_∞ on $X = \mathbb{N}$ where (x, y) is an edge of A_∞ if and only if there are naturals k, ℓ such that $x = \ell 2^k + 1$ and $y = (\ell + 1)2^k$. Unlike the path, A_∞ is not locally finite, i.e. the node 1 has infinite degree. Moreover, A_∞ is a tree.

Let F be a nonempty finite set of positive naturals and $\varphi \in S_{\mathbb{N}}$. Take $n := \lceil \log_2 \max(F \cup \varphi[F]) \rceil$.

5 Open questions

Can the Fer groups that are amenable be nicely classified the way the Aut group of trees is nicely classified?

By a result of Frucht, every finite group is the automorphism group of a regular graph, and thus, since for regular graphs the Fer group is Aut, every finite group is trivially realized as the Fer group of a finite graph. Sabidussi 1959 (https://link.springer.com/content/pdf/10.1007/B proved the analogue result for infinite groups: any group is realized as the aut group of a graph of arbitrary given cardinal. Is any infinite group realized as the Fer of some infinite graph?

References